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Viscous flow near a corner in three dimensions

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The nature of a three-dimensional viscous flow along a corner near its junction has been clarified in this paper by constructing a Stokes slow-flow solution. We have further demonstrated that this Stokes solution can be matched onto an inertial-flow solution in principle by establishing an overlap domain along one sector of an inertial-flow region, namely along the flow symmetry line. This Stokes solution reveals a remarkably complex structure of the flow as characterized by a separating streamwise velocity profile in addition to a sequence of Moffatt's viscous eddies in a cross-flow plane.

1. Introduction

The viscous flow along a right-angle corner formed by the intersection of two semi-infinite flat plates presents inherently three-dimensional characteristics, particularly near the corner, owing to the mutual interaction of the boundary layers from each plate. The flow in this geometry has been investigated by various authors as a first attempt to understand the general behaviour of a fully three-dimensional flow near corner junctions.

The first attempt on this subject by Carrier (1947) is incomplete as the streamwise vorticity has not been taken into account in his analysis. This was corrected later by Pearson (1957) and Rubin (1966), both of whom showed that a quite different cross-flow pattern indeed results with this correction. In particular, a consistent method of solution to this problem based on the method of matched asymptotic expansions was first demonstrated by Rubin (1966), who followed essentially the scheme of solution developed by Stewartson (1961) for flow past a quarter-infinite plate. In Rubin's approach the flow is divided into three regions, namely (1) the potential-flow region, (2) the boundary-layer region and (3) the corner-layer region, and solutions in each region are assumed to overlap with solutions in the other two neighbouring flow regions in appropriate intermediate domains (see also figure 1).

Despite the simple geometry involved, the flow in the corner-layer region is remarkably complex, exhibiting fully three-dimensional characteristics. It can only be determined by solving three nonlinear elliptic equations simultaneously satisfying the required asymptotic boundary conditions (see § 2 for details). Carrier (1947), Pearson (1957) and, most recently, Rubin & Grossman (1969) have all integrated the corner-layer equations numerically using a relaxation method. In particular Rubin & Grossman followed consistently the asymptotic

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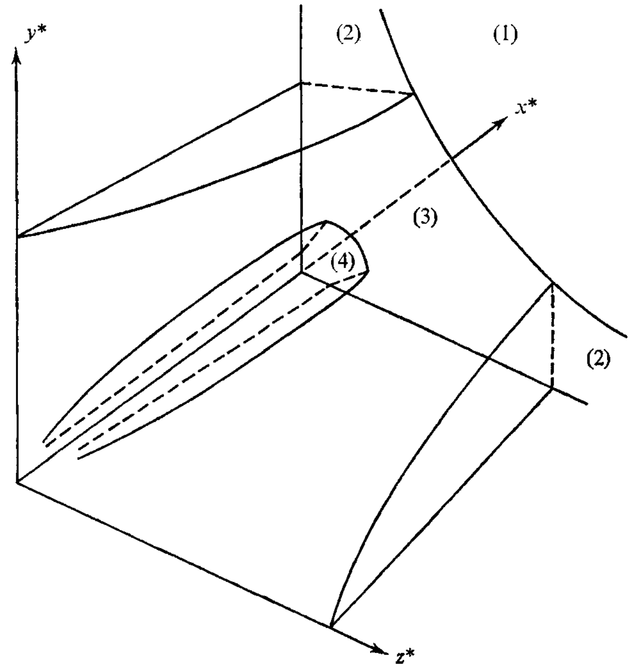

FIGURE 1. Flow past a right-angle corner and the co-ordinate system. (1) Potential-flow region, (2) boundary-layer region, (3) corner-layer region, (4) Stokes region.

matching method due to Rubin (1966). The author's attention was drawn recently to the experiments on this problem by Zamir & Young (1970) which appeared towards the end of the present investigation. It is at first very puzzling to find some considerable differences between the numerical result of Rubin & Grossman (1969) and the experiments of Zamir & Young, particularly near the corner. The differences are illustrated in figure 2. Note in particular that a considerably extended low-shear region exists along the flow symmetry line consistently in all of the experiments as the isovels have a bulge across it but the numerical solution exhibits no trace of such a bulge. Various factors seem to contribute to the complication of the flow there. The purpose of the present investigation is some elucidation of the nature of flow near the corner intersection region.

The main result presented in the present paper is two-fold. First, we have constructed a Stokes slow-flow solution for three-dimensional viscous flow near the corner region which unveils and provides clear insight into the complicated structure of the flow there. Second, we have established an overlap domain between the Stokes region and an outer inertial-flow region. Thus we have shown that the Stokes solution can be matched to an outer inertial flow as in a finite-body low Reynolds number flow. Up to now no way has been known of matching if a body involved is infinite in extent as in our problem. Actual constants have not been determined in this paper, however, because it is too difficult to obtain any explicit form of a solution in the inertial region owing to the nonlinearity of the flow. Nevertheless, a possibility for linearization in the inertial region is also discussed in § 5. The possibility of the matching is in itself important because

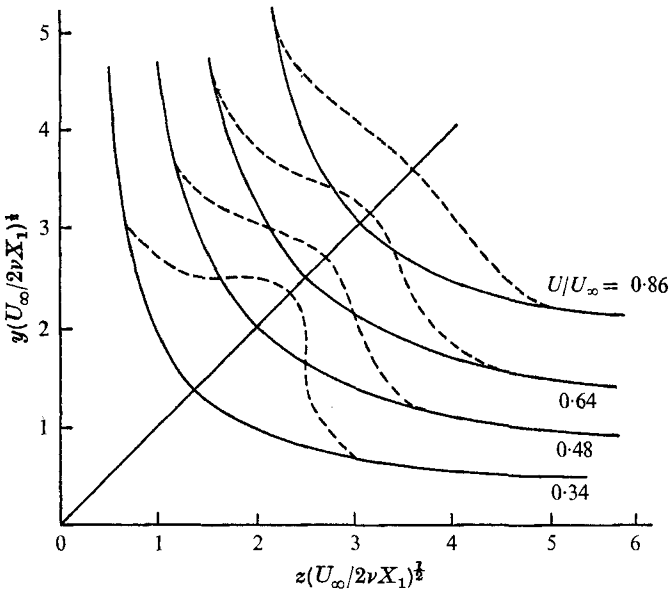

FIGURE 2. Streamwise isovels. – – –, experiment, Zamir & Young (1970); ——, numerical results, Rubin & Grossman (1969).

it now becomes clear how a 'hitherto isolated' Stokes flow joins onto an inertial-flow region comprising the dominant part of the corner-layer region.

First, we formulate the problem in § 2 using a new conformal co-ordinate system. An overlap domain is established in § 3 and the Stokes solution is obtained in § 4. An outer inertial flow is formulated in § 5.

2. Formulation

Consider a uniform incompressible viscous fluid flowing along a corner formed by two perpendicular quarter-infinite flat plates, with an undisturbed flow U at infinity parallel to the intersection of the two plates. We define Cartesian co-ordinates (x^*, y^*, z^*) along the plate with the origin at the leading edge and x^* co-ordinate along its intersection (see figure 1 for details). We shall investigate the flow on a typical characteristic plane $x = Ux^*/\nu \gg 1$.

The basic flow structure on this characteristic plane is now well understood owing largely to an excellent exposition by Rubin (1966). If $y/x \gg 1$ and $z/x \gg 1$, the flow there is irrotational and the potential-flow region (1) of figure 1 results. Here $y = Uy^*/\nu$ and $z = Uz^*/\nu$. If $y/x \gg 1$ with $z/\sqrt{x}$ fixed or $z/x \gg 1$ with $y/\sqrt{x}$ fixed, the flow approximates to the two-dimensional Blasius flow and this region (2) is called the boundary-layer region. If $y/\sqrt{x}$ and $z/\sqrt{x}$ are $O(1)$, the corner-layer region (3) results and there the flow is inherently three-dimensional owing to the mutual interaction of the boundary layers. The flow in region (3) is of the boundary-layer type in the sense that the inertial effect is of the same order as that due to diffusion. We see now that these three flow regions are still not quite general enough because another singular flow region exists and is in fact imbedded within region (3). In this region Stokes slow-flow motion dominates

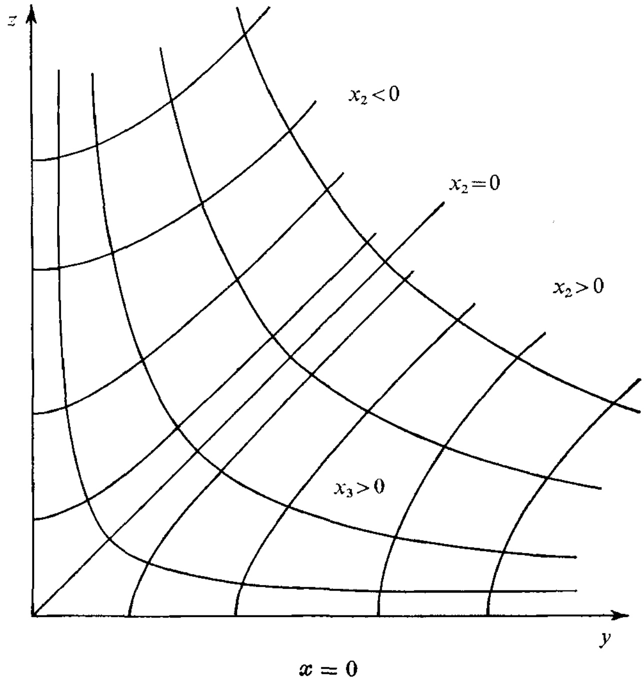

FIGURE 3. Conformal co-ordinate system.

and this region, with which our present analysis will be mainly concerned, will be called the inner (Stokes) corner-layer region (4). This inner corner-layer region results if y/z is fixed as $y/\sqrt{x} \to 0$, i.e. $y/\sqrt{x}$ and $z/\sqrt{x} \to 0$ simultaneously at the same speed. From this physical background it is clear that we must seek a solution for the corner-layer region (3) such that it will join smoothly onto solutions valid not only in (1) and (2) respectively, but also onto a Stokes solution in (4) which we shall construct in this paper. Because the Stokes region (4) is a singular region, much care seems to be required in order to extend the corner-layer equation by integration into the Stokes region. It is possible that the disagreement of Rubin & Grossman's (1969) numerical solution with the experiment of Zamir & Young (1970) is partly due to a failure, on the part of the numerical computations, to account for the proper match with the inner corner-layer solution (4).

In the analysis of the corner-layer flow all the investigators on the subject (Carrier 1947; Pearson 1957; Rubin 1966, etc.) used Cartesian co-ordinates. Although the Cartesian co-ordinates have a definite advantage of simplicity, and also of smooth join into the boundary-layer region (2),† they are not natural co-ordinates for the problem. One of the simpler sets of natural co-ordinates for this problem‡ can be found using a simple conformal transformation as follows:

$$x_1 = x, \quad x_2 + ix_3 = (z+iy)^2. \tag{2.1}$$

† The Blasius variable η_B is given as $\eta_B = y/\sqrt{x}$ with $z/\sqrt{x} \gg 1$. Hence η_B is obtained directly from Cartesian co-ordinates on a characteristic surface of $x =$ constant $\gg 1$.

‡ Zamir (1970) suggested another set of natural co-ordinates to this problem: $x_1 = x$, $x_2 = (y^3 - z^3)^{\frac{1}{3}}$, $x_3 = yz/(y+z)$. Zamir's co-ordinates are too complicated to use, particularly in transforming the biharmonic equations.

We note that in this system the line of flow symmetry is described by $x_2 = 0$ and the plate wall by $x_3 = 0$ everywhere (see figure 3). Far from the corner, as $x_2 \to \infty$, some stretching of variables is necessary to obtain the Blasius variable (see equation (2.22) for example).

Cylindrical polar co-ordinates are also useful in this problem although they are not natural co-ordinates in the sense used above.

$$\xi = x, \quad re^{i\theta} = z + iy. \tag{2.2}$$

Without exception, existing two-dimensional Stokes-flow solutions near corners (Carrier & Lin 1948; Moffatt 1964; Lugt & Schwiderski 1964) have been obtained in polar co-ordinates. It is often true that some flow properties as well as some algebra required in the analysis are greatly simplified using polar co-ordinates (see equation (4.11), for example). On the other hand, the use of the conformal co-ordinates (2.1) is crucial in establishing an overlap domain from the Stokes to the inertial-flow region as we shall discuss in detail in § 3, and no way of establishing this using the polar co-ordinates of (2.2) seems to be known.

The complete Navier–Stokes equations can be expressed in terms of the new orthogonal co-ordinate system of (2.1) following the usual method of orthogonal vector geometry in which only the diagonal metric tensors remain. In particular, the formulae given by Lagerstrom (1964, p. 60) are most useful in working out the details of the present transformation. The final results are

$$\frac{\partial}{\partial x_1}\left(\frac{v_1}{J^*}\right) + \frac{\partial}{\partial x_2}\left(\frac{v_2}{J^{*\frac{1}{2}}}\right) + \frac{\partial}{\partial x_3}\left(\frac{v_3}{J^{*\frac{1}{2}}}\right) = 0, \tag{2.3}$$

$$v_1 \frac{\partial v_1}{\partial x_1} + J^{*\frac{1}{2}}\left(v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}\right) = \frac{\partial P}{\partial x_1} + \nabla^{*2} v_1, \tag{2.4}$$

$$\begin{aligned} v_1 \frac{\partial v_2}{\partial x_1} + J^{*\frac{1}{2}}\left(v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}\right) + \frac{8}{J^{*\frac{3}{2}}}(x_2 v_3^2 - x_3 v_2 v_3) \\ = J^{*\frac{1}{2}} \frac{\partial P}{\partial x_2} + \nabla^{*2} v_2 + \frac{16}{J^*}\left(x_2 \frac{\partial v_3}{\partial x_3} - x_3 \frac{\partial v_3}{\partial x_2} - \frac{v_2}{4}\right), \end{aligned} \tag{2.5}$$

$$\begin{aligned} v_1 \frac{\partial v_3}{\partial x_1} + J^{*\frac{1}{2}}\left(v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}\right) + \frac{8}{J^{*\frac{3}{2}}}(x_3 v_2^2 - x_2 v_2 v_3) \\ = J^{*\frac{1}{2}} \frac{\partial P}{\partial x_3} + \nabla^{*2} v_3 + \frac{16}{J^*}\left(x_3 \frac{\partial v_2}{\partial x_3} - x_2 \frac{\partial v_2}{\partial x_3} - \frac{v_3}{4}\right). \end{aligned} \tag{2.6}$$

Here
$$J^* = 4(x_2^2 + x_3^2)^{\frac{1}{2}}, \quad \nabla^{*2} = \frac{\partial^2}{\partial x_1^2} + J^*\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \tag{2.7}$$

and (v_1, v_2, v_3) are the dimensionless velocity components corresponding to (x_1, x_2, x_3) respectively. We particularly note that J^* is independent of the x_1 co-ordinate.

Following Rubin (1966), the governing equations in the potential-flow, boundary-layer and corner-layer regions can be deduced from these Navier–Stokes equations. We shall merely derive the governing equations for the corner-layer region from which the equations for a Stokes region also result. The derivations in the other two regions are similar but will not be given here.

First, we introduce the following corner-layer variables:

$$X_1 = x_1, \quad X_2 = x_2/X_1, \quad X_3 = x_3/X_1. \tag{2.8}$$

The appropriate corner-layer limit is X_2 and X_3 fixed as $X_1 \to \infty$. A consistent asymptotic expansion in the corner-layer region can be obtained by applying the corner-layer limit as follows (see also Rubin 1966):

$$\left.\begin{aligned}
v_1 &= u(X_1, X_2, X_3) = u_0(X_2, X_3) + X_1^{-\frac{1}{2}} u_1(X_2, X_3) + X_1^{-1} u_2(X_2, X_3) + \ldots, \\
v_2 &= v(X_1, X_2, X_3) = X_1^{-\frac{1}{2}} v_1(X_2, X_3) + X_1^{-1} v_2(X_2, X_3) + \ldots, \\
v_3 &= w(X_1, X_2, X_3) = X_1^{-\frac{1}{2}} w_1(X_2, X_3) + X_1^{-1} w_2(X_2, X_3) + \ldots, \\
P &= P(X_1, X_2, X_3) = P_0(X_2, X_3) + X_1^{-\frac{1}{2}} P_1(X_2, X_3) + X_1^{-1} P_2(X_2, X_3) + \ldots.
\end{aligned}\right\} \tag{2.9}$$

By substituting (2.9) into (2.3)–(2.7), the leading terms of the expansion for the corner-layer region can be obtained as follows. For simplicity, we drop the subscripts from u_0, v_1, w_1 and P_2 hereafter. Now,

$$-X_2\frac{\partial u}{\partial X_2} - X_3\frac{\partial u}{\partial X_3} + J\frac{\partial}{\partial X_2}\left(\frac{v}{J^{\frac{1}{2}}}\right) + J\frac{\partial}{\partial X_3}\left(\frac{w}{J^{\frac{1}{2}}}\right) = 0, \tag{2.10}$$

$$-u\left(X_2\frac{\partial u}{\partial X_2} + X_3\frac{\partial u}{\partial X_3}\right) + J^{\frac{1}{2}}\left(v\frac{\partial u}{\partial X_2} + w\frac{\partial u}{\partial X_3}\right) = J\left(\frac{\partial^2 u}{\partial X_2^2} + \frac{\partial^2 u}{\partial X_3^2}\right), \tag{2.11}$$

$$\begin{aligned}
-u\left(\frac{v}{2} + X_2\frac{\partial v}{\partial X_2} + X_3\frac{\partial v}{\partial X_3}\right) &+ J^{\frac{1}{2}}\left(v\frac{\partial v}{\partial X_2} + w\frac{\partial v}{\partial X_3}\right) + \frac{8}{J^{\frac{3}{2}}}(X_2 w^2 - X_3 vw) \\
&= J^{\frac{1}{2}}\frac{\partial P}{\partial X_2} + J\left(\frac{\partial^2 v}{\partial X_2^2} + \frac{\partial^2 v}{\partial X_3^2}\right) + \frac{16}{J}\left(X_2\frac{\partial w}{\partial X_2} - X_3\frac{\partial w}{\partial X_3} - \frac{v}{4}\right),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
-u\left(\frac{w}{2} + X_2\frac{\partial w}{\partial X_2} + X_3\frac{\partial w}{\partial X_3}\right) &+ J^{\frac{1}{2}}\left(v\frac{\partial w}{\partial X_2} + w\frac{\partial w}{\partial X_3}\right) + \frac{8}{J^{\frac{3}{2}}}(X_3 v^2 - X_2 vw) \\
&= J^{\frac{1}{2}}\frac{\partial P}{\partial X_3} + J\left(\frac{\partial^2 w}{\partial X_2^2} + \frac{\partial^2 w}{\partial X_3^2}\right) + \frac{16}{J}\left(X_3\frac{\partial v}{\partial X_3} - X_2\frac{\partial v}{\partial X_2} - \frac{w}{4}\right),
\end{aligned} \tag{2.13}$$

where $J = 4(X_2^2 + X_3^2)^{\frac{1}{2}}$ and we note that

$$\frac{\partial P_0}{\partial X_2} = \frac{\partial P_0}{\partial X_3} = \frac{\partial P_1}{\partial X_2} = \frac{\partial P_1}{\partial X_3} = 0.$$

The continuity equation (2.10) can be satisfied exactly by introducing the two vector potential functions ψ and ϕ such that

$$\frac{u}{J} = \psi_{X_3}, \quad \frac{v}{J^{\frac{1}{2}}} = \phi_{X_3}, \quad \frac{w}{J^{\frac{1}{2}}} = X_2\psi_{X_2} + X_3\psi_{X_3} - \phi_{X_2}. \tag{2.14}$$

It is convenient to rewrite (2.11)–(2.13) as follows:

$$L_1(u) = N_1(u, v, w), \tag{2.15}$$

$$L_2(\phi) = H(\psi) + N_2(u, v, w), \tag{2.16}$$

in which the velocity field (u, v, w) is related to ψ and ϕ by (2.14). L_1 and L_2 are harmonic and biharmonic operators and in the present co-ordinate system are given by

$$L_1 = J\nabla^2, \quad L_2 = J\nabla^2(J\nabla^2), \tag{2.17, 2.18}$$

where $\nabla^2 = \partial^2/\partial X_2^2 + \partial^2/\partial X_3^2$ and $J = 4(X_2^2+X_3^2)^{\frac{1}{2}}$. N_1 is the operator involving the nonlinear convective terms, namely those on the left-hand side of (2.11); (2.16) is obtained from (2.12) and (2.13) by eliminating pressure by cross-differentiation. H is a fourth-order operator like the biharmonic operator L_2 of (2.16) and arises through the interaction of ψ in the w velocity component (see (2.14), for example):

$$H(\psi) = J\frac{\partial}{\partial X_3}\left[\frac{16}{J}\left\{X_2\frac{\partial}{\partial X_3}(X_2\psi_{X_2}+X_3\psi_{X_3}) - X_3\frac{\partial}{\partial X_2}(X_2\psi_{X_2}+X_3\psi_{X_3})\right\}\right]$$
$$+J\frac{\partial}{\partial X_2}\left[\frac{4(X_2\psi_{X_2}+X_3\psi_{X_3})}{J} - J^{\frac{1}{2}}\nabla^2\{J^{\frac{1}{2}}(X_2\psi_{X_2}+X_3\psi_{X_3})\}\right]. \quad (2.19)$$

N_2 is similar to N_1 of (2.15) and involves the nonlinear convective terms. It can be formally obtained by cross-differentiating (2.12) and (2.13) and taking their differences.

The appropriate boundary conditions are

$$\phi = \frac{\partial\phi}{\partial X_3} = \psi = \frac{\partial\psi}{\partial X_3} = 0, \quad \text{at} \quad X_3 = 0 \text{ for all } X_2; \quad (2.20)$$

$$(\phi,\psi) \to (\phi_P,\psi_P) \quad \text{as} \quad X_3 \to \infty \text{ with } X_2 \text{ fixed}, \quad (2.21)$$

$$(\phi,\psi) \to (\phi_B,\psi_B) \quad \text{as} \quad X_2 \to \infty \text{ with } X_3/X_2 \text{ fixed},\dagger \quad (2.22)$$

where (ϕ_P,ψ_P) and (ϕ_B,ψ_B) are the velocity components in the potential-flow and boundary-layer regions respectively (see Pal & Rubin (1969) for a detailed asymptotic analysis). For the complete specification of the boundary conditions for this elliptic problem one needs another condition along $X_2 = 0$ for any X_3. This can in principle be satisfied by the flow symmetry condition. Obviously u and w are symmetric in X_2 and v is anti-symmetric in X_2:

$$u(X_2,X_3) = u(-X_2,X_3),$$
$$v(X_2,X_3) = -v(-X_2,X_3),$$
$$w(X_2,X_3) = w(-X_2,X_3).$$

In terms of the vector potential functions ϕ and ψ, this symmetry condition requires

$$\psi(X_2,X_3) = \psi(-X_2,X_3), \quad \phi(X_2,X_3) = -\phi(-X_2,X_3). \quad (2.23)$$

In principle, (2.15)–(2.23) are sufficient to determine a complete flow in the corner-layer region. When the flow in the corner-layer region as fully described above is examined closely, it becomes quite clear that two distinct flow regions, namely an inner Stokes-flow domain and an outer inertial-flow domain, co-exist in the corner-layer region. In the Stokes approximation, which is valid sufficiently close to the sharp corner, the diffusion dominates the convection. Therefore, the problem may be linearized here. Away from the corner, we enter an inertial-flow region in which the effects of diffusion and convection are of the same order.

† This restretching is necessary if the present co-ordinate system is to asymptote to the Blasius-type boundary-layer region appropriately. This is similar to the derivation of the Blasius variable for flow past a semi-infinite flat plate in Cartesian co-ordinates.

In § 3, we demonstrate consistent inner and outer limiting processes which provide a Stokes and an inertial-flow expansion, as described above, from the corner-layer equations (2.15) and (2.16) (see equations (3.1) and (3.2) for details). Most important, an overlap domain between the inner Stokes flow and the outer inertial-flow domains will be demonstrated. All of the previous works on the subject (Carrier 1947; Pearson 1957; Rubin 1966; Rubin & Grossman 1969) have emphasized the inertial-flow domain and little attention has been paid to the singular Stokes domain.

3. Inner and outer limit and overlap domain

A consistent inner expansion, a solution of which reduces to that of Stokes slow-flow motion, can be constructed formally by introducing the following inner variables:

$$Y = X_3/X_2 \quad \text{fixed as} \quad X_2 \to 0. \tag{3.1}$$

In this inner limit we note that $X_3 \to 0$ since the inner variable Y remains fixed. An inner Stokes solution obtained by this limiting process shows that the diffusion dominates the convection, allowing the inner expansion in the form of (4.1) or (4.2) (see § 4). If we keep X_3 fixed as $X_2 \to 0$, the inner variable $Y \to \infty$. This suggests that another distinct flow region exists under such a limiting process. We should, therefore, now introduce the following outer variables in the domain where $Y \to \infty$:

$$\eta = X_3 \quad \text{fixed as} \quad X_2 \to 0. \tag{3.2}$$

This limit is called an outer limit and it is easy to confirm that the inertial term and the diffusion term obtained in this limiting process balance in this domain. (See equation (5.5) of § 5 for further clarification of this point.) Because the inertial-flow region in fact extends into the corner-flow region for any finite X_2 and X_3, (3.2) describes only one sector of the complete inertial-flow region, namely that along the flow symmetry line. We shall now prove that an inner Stokes solution obtained by the limiting process of (3.1) and an outer inertial flow solution can be matched along this particular sector near the flow symmetry line of the inertial region as described by (3.2).

To prove this, we first introduce the following intermediate limit:

$$Y_I = X_3/X_2^{\alpha} \quad \text{fixed as} \quad X_2 \to 0 \tag{3.3}$$

with $0 < \alpha < 1$. By comparing (3.1) and (3.3) and introducing them into (2.15) and (2.16), we can easily confirm in both (3.1) and (3.3) that

$$L_i \gg N_i \quad \text{for} \quad i = 1, 2.$$

This implies, in both the Stokes and intermediate limits, that the diffusion effect dominates the inertial effect. Unlike the finite-body analysis of, say, Proudman & Pearson (1957), in which a small parameter exists, the inner and intermediate problems of (3.1) and (3.3) do not reduce to the same problem because a small co-ordinate X_2 rather than a parameter is used in the stretching. It is not difficult to see, however, that an intermediate problem can be deduced from an inner problem by considering the limit of $Y \to \infty$ in the inner limit. This means that the inner Stokes solution is the more general solution of the two and overlaps with an intermediate solution over the complete domain defined by $0 < \alpha < 1$. By now comparing (3.2) and (3.3), we see that an overlap domain exists between the outer inertial region and the intermediate region, hence the inner Stokes region can be established provided that the validity of the outer solution can be shown to extend even slightly towards the inner region. This extension of the validity of an outer solution is assured by Kaplun's extension theorem, which can be directly applied to this problem (see Kaplun 1967, p. 97). We have thus proved that an overlap domain exists between the inner Stokes solution and the inertial-flow solution along $X_2 \to 0$. This proves that an inner Stokes solution and an inertial solution can be matched along $X_2 \to 0$ in this problem. In the present analysis, we merely follow a conventional general matching principle, namely

$$\begin{aligned}&\text{The outer representation of the inner expansion (as } Y \to \infty)\\ &\quad = \text{the inner representation of the outer expansion (as } \eta \to 0). \qquad (3.4)\end{aligned}$$

One of the great advantages in using the present conformal co-ordinate system is that one can thus define the inner and outer limit and carry out the conventional matching procedure of the asymptotic expansions most systematically. As a check, one can construct an inner Stokes expansion using other co-ordinate systems. The most well-known one is the polar co-ordinate system used by Carrier & Lin (1948) and others for two-dimensional flows. The corresponding inner limit to (3.1) will then be

$$\theta \quad \text{fixed as} \quad r \to 0, \qquad (3.5)$$

where r, θ are polar co-ordinates. In this approach, we have no means of determining how an inner solution constructed by the limiting process of (3.5) can join onto an outer flow as we have clearly demonstrated in (3.1)–(3.4). Our preceding discussion shows that an inner Stokes solution can be matched onto an outer inertial solution in the following region in polar co-ordinates:

$$r \quad \text{finite as} \quad \theta \to 0. \qquad (3.6)$$

If we are to stick with polar co-ordinates in the analysis, it is not at all clear that an inner solution of (3.5) and an outer solution of (3.6) will ever match. It is owing to the difficulty of establishing an overlap domain in polar co-ordinates that no progress has been made in the analysis beyond the construction of an inner Stokes solution. We are now certain that an inner Stokes solution matches with an inertial-flow solution in the overlap domain† of (3.2) or (3.6). A difficulty in determining all the constants of a Stokes-flow solution remains here, however, because it seems too difficult to obtain an explicit analytic solution for the outer inertial-flow region. A complete numerical integration in the inertial-flow region seems necessary, with the outer form of a Stokes solution providing the boundary conditions on $X_2 = 0$. We give further detailed discussion on this point in § 5. We shall now present an inner Stokes solution.

† A similar situation arises in viscous flow near the leading edge of a semi-infinite flat plate or, for that matter, in any two-dimensional viscous flow near corners past a semi-infinite body. I have already given a detailed explanation in the former leading-edge flow why an overlap domain may not be established using polar co-ordinates (Tokuda 1972).

4. Inner Stokes-flow solution

An inner Stokes expansion can be obtained from the corner-layer equations (2.15) and (2.16) by repeated applications of the inner limit of (3.1). Despite the enormously complicated form of governing equations (2.15) and (2.16), a remarkably simple scheme of solutions emerges by this inner limiting process. We shall show that a self-consistent inner Stokes solution for ψ and ϕ can be given by the following asymptotic expansions:

$$\begin{aligned}\psi(X_2, Y) &\sim \psi_1+\psi_{30}+\psi_{31}+\dots \\ &\sim X_2 G_1(Y)+X_2^3 \ln X_2 G_{30}(Y)+X_2^3 G_{31}(Y)+\dots, \end{aligned} \tag{4.1}$$

$$\begin{aligned}\phi(X_3, Y) &\sim \phi_{20}+\phi_{\lambda_1}+\phi_{30}+\phi_{31}+\dots \\ &\sim X_2^2 K_{20}(Y)+X_2^{\lambda_1} K_{\lambda_1}(Y)+X_2^3 \ln X_2 K_{30}(Y)+X_2^3 K_{31}(Y)+\dots, \end{aligned} \tag{4.2}$$

all with Y fixed as $X_2 \to 0$. λ_1 is a complex number given by Moffatt (1964) as

$$\lambda_1 = 2{\cdot}904+0{\cdot}7322i.$$

From the computational point of view, the streamwise velocity u is often more convenient than ψ in this problem because u is governed by harmonic operators in the inner region as we shall soon see. Because ψ and u are related by (2.14) the inner expansion for u is given as

$$\begin{aligned}u(X_2, Y) &\sim u_1+u_{30}+u_{31}+\dots \\ &\sim X_2 F_1(Y)+X_2^3 \ln X_2 F_{30}(Y)+X_2^3 F_{31}(Y)+\dots, \end{aligned} \tag{4.3}$$

with

$$G_N(Y) = \frac{1}{4}\int \frac{F_N(Y)}{(1+Y^2)^{\frac{1}{2}}}\,dY.$$

Appropriate boundary conditions which one can specify for the inner Stokes solution are the no-slip flow condition of (2.20) and the flow symmetry condition of (2.23). Boundary conditions in a far field, namely (2.21) and (2.22), cannot be enforced in this Stokes solution and must be replaced by the matching condition (3.4) to the outer inertial-flow solution.

The basic scheme of solution in the inner Stokes flow becomes evident by substituting the expansion of (4.1)–(4.3) into (2.15) and (2.16). First, we note that the nonlinear terms N_1 and N_2 of (4.1) and (4.2) are of higher orders† than those of harmonic or biharmonic terms on the left-hand side of those equations. Therefore, the streamwise velocity u is basically governed by harmonic equations and the cross-flow component ϕ by biharmonic equations.

4.1. *First-order streamwise flow solution u_1 and ψ_1*

By substituting (4.3) into (2.15), we see that F_1 must satisfy

$$(1+Y^2)F_1'' = 0. \tag{4.4}$$

† For example, N_1 is definitely of higher order than $L_1(u)$ unless v, w are of $O(X_2^{-\frac{1}{2}})$. This obviously cannot be true in the present situation because, if it were so, one would be able to find a biharmonic solution $\nabla^4\phi = 0$ in the form $\phi = X_2^{\frac{1}{2}}G(Y)$, and Moffatt (1964) shows that this is not possible.

A solution which satisfies the required boundary conditions is

$$F_1 = A_1 Y \quad \text{or} \quad u_1 = A_1 X_2 Y. \tag{4.5}$$

In an outer variable, this gives $u_1 = A_1 X_3$. Hence

$$G_1 = \tfrac{1}{4}A_1[(1+Y^2)^{\frac{1}{2}}-1]\,\mathrm{sgn}\,Y \quad \text{or} \quad \psi_1 = \tfrac{1}{4}A_1X_2[(1+Y^2)^{\frac{1}{2}}-1]\,\mathrm{sgn}\,Y, \tag{4.6}$$

where A_1 is an arbitrary constant.

4.2. *First-order cross-flow solution ϕ_2*

Now ψ_1 of (4.6) introduces a non-homogeneous term into the cross-flow equation (2.16) through the first term of the right-hand side, $H(\psi_1)$. Substituting ψ_1 into (2.19), we find that

$$H(\psi_1) = -4A_1\,\mathrm{sgn}\,Y. \tag{4.7}$$

Therefore, we must expect a term of $O(X_2^{\frac{3}{2}})$ in the cross-flow component ϕ as follows. We write

$$\phi_2 = X_2^{\frac{3}{2}}K_2(Y)\,\mathrm{sgn}\,Y, \tag{4.8}$$

then K_2 must satisfy

$$(1+Y^2)^3K_2^{\mathrm{iv}} + 4Y(1+Y^2)^2K_2''' + (1+Y^2)K_2'' - 2YK_2' + 2K_2 = -\tfrac{1}{4}A_1. \tag{4.9}$$

A solution for K_2 which satisfies the boundary condition (2.20) and the flow symmetry condition (2.23) is

$$K_2 = \tfrac{1}{8}A_1[(1+Y^2)^{\frac{1}{2}}-1]. \tag{4.10a}$$

Hence

$$\phi_2 = \tfrac{1}{8}A_1X_2^{\frac{3}{2}}[(1+Y^2)^{\frac{1}{2}}-1]\,\mathrm{sgn}\,Y; \tag{4.10b}$$

ϕ_2 gives the cross-flow components

$$v_2 = \tfrac{1}{4}A_1X_2^{\frac{3}{2}}Y/(1+Y^2)^{\frac{1}{2}} \quad \text{and} \quad w_2 = \tfrac{1}{4}A_1X_2^{\frac{3}{2}}\,Y^2/(1+Y^2)^{\frac{1}{2}}.$$

It is worthwhile noting that in cylindrical polar co-ordinates the first-order velocity of (4.5) and (4.10) above represents a much simpler radial flow field given by

$$u_1 = A_1r^2\sin 2\theta, \quad v_{r_2} = \tfrac{1}{4}A_1r^3\sin 2\theta, \quad v_{\theta_2} = 0; \tag{4.11}$$

v_r and v_θ denote the velocity component in the r and θ directions respectively. In polar co-ordinates only the radial component v_r exists in the cross flow. Physically this v_r is merely the outflow necessary to balance the variation of u along the main-stream direction.

4.3. '*Viscous-eddy*' *cross-flow solution K_{λ_1} or ϕ_{λ_1}*

Using N_2 of (2.16) and ϕ_2 above, it is found that the nonlinear effect in this problem starts to appear at $O(X_3^{\frac{3}{2}})$ or $O(r^6)$. One of the most interesting results in the present three-dimensional corner-flow problem is the appearance of viscous eddies in the cross-flow plane preceding this nonlinear effect. It was Moffatt (1964) who first correctly noted and interpreted the appearance of viscous eddies consisting of a sequence of eddies of rapidly decreasing size and intensity near a sharp two-dimensional corner if its angle is less than about 156° for symmetrical flows.

Write

$$\phi_{\lambda_1} = X_2^{\lambda_1}K_{\lambda_1}(Y), \tag{4.12}$$

then, assuming that $\operatorname{Re}\{\lambda_1\} < 3{\cdot}0$, ϕ_{λ_1} must satisfy the homogeneous biharmonic equations:

$$L_2(\phi_{\lambda_1}) = 0, \tag{4.13}$$

where L_2 is a biharmonic operator given in (2.18). The equation for ϕ_{λ_1} reduces exactly to a Stokes solution for a corner in two dimensions. Because the corner angle 2α of the present problem is $90°$ and is less than a critical angle $156°$, a viscous-eddy solution of Moffatt's type must arise in this flow. Biharmonic solutions to (4.13) can in principle be obtained by the present conformal co-ordinates. The process of finding eigenvalues for the exponent becomes very tedious in this co-ordinate, however, because a pair of fourth-order differential equations must be solved simultaneously such that the flow symmetry condition results for $Y \to \infty$. In polar co-ordinates, these complex eigenvalues can be found by merely solving a simple transcendental equation. In terms of the polar co-ordinates, Moffatt (1964) shows the leading term for $2\alpha = 90°$ for symmetrical flow is

$$\phi_{\lambda_1} = r^{2\lambda_1}(B_1 \sin \lambda_1 \theta + C_1 \sin(\lambda_1 - 2)\theta), \tag{4.14}$$

where $2\lambda_1 = \kappa_1 + i\kappa_2$ with $\kappa_1 = 5{\cdot}80825$, $\kappa_2 = 1{\cdot}4639$ and B_1 and C_1 some arbitrary constants. r and θ are related to the present inner variables Y and X_2 by

$$r = X_2^{\frac{1}{2}}(1+Y^2)^{\frac{1}{4}}, \quad 2\alpha = \tan^{-1} 1/Y. \tag{4.15}$$

Equation (4.14) can always be rewritten as (Moffatt 1964)

$$\phi_{\lambda_1} = A_1 r^{\kappa_1} \sin(\kappa_2 \ln r + \epsilon), \tag{4.16}$$

ϵ and A_1 being functions of θ. Equation (4.16) shows clearly how viscous eddies must result as $r \to 0$. The outer limit of (4.14) in the present limit of (3.2) can be obtained by choosing $\theta \to 0$. Accordingly, the outer representation of ϕ_{λ_1} is

$$\phi_{\lambda_1} = \tfrac{1}{2} D_1 \eta^{\frac{1}{2}(\kappa-1)} \sin(\kappa_2 \ln \eta + \epsilon_0) + O(\eta^3), \tag{4.17}$$

where

$$D_1^2 = (B_1 + C_1)^2(\kappa_1^2 + \kappa_2^2) - 4C_1\kappa_1(B_1 + C_1) + 4C_1^2 \quad \text{and} \quad \epsilon_0 = \cos^{-1}((B_1 + C_1)\kappa_2/D_1).$$

Equation (4.17) is an outer form of the leading term of the viscous-eddy solution.

4.4. *Second-order streamwise flow u_3 and ψ_3*

Intuition suggests that the next term of the streamwise velocity u_3 is $O(X_2^{\frac{3}{2}})$, but the outer expansion of the inner solution u_3, namely X_3 fixed as $X_2 \to 0$, develops a discontinuity along the line of flow symmetry $X_2 = 0$. (See equation (4.25) for clarification of this point.) This cannot be physically accepted. A correct solution must be preceded by a logarithmic term as is so often the case with singular perturbation problems. We therefore write

$$u = X_2 F_1(Y) + X_2^{\frac{3}{2}} \ln X_2 F_{30}(Y) + X_2^{\frac{3}{2}} F_{31}(Y) + \dots; \tag{4.18}$$

F_{30} must satisfy

$$(1+Y^2)F_{30}'' - 4YF_{30}' + 6F_{30} = 0. \tag{4.19}$$

A solution for F_{30} is

$$F_{30} = A_{30}(Y - \tfrac{1}{3}Y^3), \tag{4.20}$$

where A_{30} is an arbitrary constant which must be chosen such that a discontinuity of velocity will disappear, as we demonstrate later.

F_{31} must now satisfy

$$(1+Y^2)F''_{31}-4YF'_{31}+6F_{31} = -A_{30}(\tfrac{1}{3}Y^3+3Y)-\operatorname{sgn}(Y)A_1^2Y^2/8(1+Y^2)^{\frac{1}{2}}. \quad (4.21)$$

We note that sgn (Y) in the last term is necessary because the absolute value is always understood for the Jacobian J. Note that in terms of the inner variable J becomes

$$J = 4(X_2^2+X_3^2)^{\frac{1}{2}} = 4\operatorname{sgn}(X_2)\times X_2(1+Y^2)^{\frac{1}{2}} = 4X_2\operatorname{sgn}(Y)(1+Y^2)^{\frac{1}{2}}.$$

We now write

$$F_{31} = F_{P_1}+F_{P_2}+F_C, \quad (4.22)$$

$$L(F_{P_1}) = \frac{A_1^2}{8}\frac{y^2}{(1+Y^2)^{\frac{1}{2}}}\operatorname{sgn} Y, \quad (4.23)$$

$$L(F_{P_2}) = -A_{30}(\tfrac{1}{3}Y^3+3Y), \quad (4.24)$$

where $L = (1+Y^2)\,d^2/dY^2-4Y\,d/dY+6$ and F_C is a complementary solution.

Two independent complementary functions to $L(F) = 0$ are $(Y-\frac{1}{3}Y^3)$ and $(1-3Y^2)$; F_{P_1} and F_{P_2} can then be obtained by the method of variation of parameters. The final results are

$$F_{P_1} = \tfrac{1}{360}A_1^2\operatorname{sgn}(Y)\,[(1+Y^2)^{\frac{1}{2}}(2-7Y^2)-2(1-3Y^2)], \quad (4.25)$$

$$F_{P_2} = A_{30}[\tfrac{1}{3}(1-3Y^2)\tan^{-1}Y+\tfrac{1}{6}(3Y-Y^3)\ln(1+Y^2)+\tfrac{1}{9}Y(Y^2-3)]. \quad (4.26)$$

We immediately note that the outer form of the inner solution with F_{P_1} alone (that is, without the preceding logarithmic term) contains a discontinuity across $X_2 = 0$ with X_3 finite as

$$F_{P_1} \sim -\tfrac{1}{180}A_1^2+O(X_3) \quad \text{at} \quad X_2 = 0^+,$$
$$\sim \tfrac{1}{180}A_1^2+O(X_3) \quad \text{at} \quad X_2 = 0^-.$$

Such a discontinuity cannot be allowed in the solution. However, this can be avoided and hence the present asymptotic expansion can be continued without failure if we include the logarithmic term and choose the constant A_{30} as

$$A_{30} = A_1^2/30\pi. \quad (4.27)$$

A complete solution for F_{31} is therefore

$$F_{31} = A_{31}(Y-\tfrac{1}{3}Y^3)+\tfrac{1}{360}A_1^2[\operatorname{sgn}(Y)\{(1+Y^2)^{\frac{1}{2}}(2-7Y^2)-2(1-3Y^2)\}$$
$$+(6/\pi)\{\tfrac{1}{3}(1-3Y^2)\tan^{-1}Y+\tfrac{1}{6}(3Y-Y^3)\ln(1+Y^2)+\tfrac{1}{9}(Y^3-3Y)\}], \quad (4.28)$$

where A_{31} is an arbitrary constant. For $Y \ll 1$ and $Y \gg 1$, F_{31} has the following expansions:

$$F_{31} \sim A_{31}Y-\frac{Y^3}{3}\left(A_{31}+\frac{A_1^2}{10\pi}\right)-\frac{A_1^2Y^4}{96}-\frac{A_1^2}{360\pi}Y^5+\dots \quad \text{for} \quad Y \ll 1, \quad (4.29)$$

$$F_{31} \sim -\frac{Y^3\ln Y}{45\pi}+\frac{Y^3}{3}\left\{-A_{31}+\frac{A_1^2}{360}\left(\frac{4}{\pi}-21\right)\right\}+\frac{A_1^2Y\ln Y}{30\pi}$$
$$+Y\left\{A_{31}-\frac{A_1^2}{360}\left(\frac{16}{\pi}+\frac{3}{2}\right)\right\}-\frac{A_1^2}{40}\left(\frac{1}{\pi}-\frac{1}{8}\right)\frac{1}{Y}+O\left(\frac{1}{Y^3}\right) \quad \text{for} \quad Y \gg 1. \quad (4.30)$$

4.5. *Second-order cross-flow solution* ϕ_{30}, ϕ_{31}

The second-order streamwise flow ψ_{30} and ψ_{31} again induces essentially through the continuity equation the cross-flow velocity, because then $H(\psi_{30})$ and $H(\psi_{31})$ do not vanish. In addition, the nonlinear term N_2 starts to have an effect in this problem. Therefore we write

$$\phi = X_2^2 K_2(Y) + X_2^{\lambda_1} K_{\lambda_1}(Y) + X_2^3 \ln X_2 K_{30}(Y) + X_2^3 K_{31}(Y) + \dots . \tag{4.31}$$

K_{30} and K_{31} must now satisfy the following non-homogeneous equations:

$$L_3(K_{30}) = C_{30}, \tag{4.32}$$

$$L_3(K_{31}) = C_{31}, \tag{4.33}$$

where
$$L_3 = (1+Y^2)\frac{d^4}{dY^4} + 3(1+Y^2)\frac{d^2}{dY^2} - 6Y\frac{d}{dY} + 18 \tag{4.34}$$

and C_{30} and C_{31} are non-homogeneous terms. The C_{30} term arises from $H(\psi_{30})$ and N_2 of (2.16). The contribution to C_{31} arises from the terms $H(\psi_{30}+\psi_{31})$, N_2 and also K_{30}. Neither C_{30} nor C_{31} seem to vanish but evaluation of these terms has not been done in the present paper as the algebra seems too tedious, owing partly to the complicated form of ψ_{31} and also the operator H (see equation (2.19)).

The present scheme of solution can in principle be continued to higher-order expansions. The next term of the inner solution for the stream-wise velocity field does arise from the viscous-eddy cross-flow solution ϕ_{λ_1}. The expansion, therefore, looks like

$$u = X_2 F_1(Y) + X_2^3 \ln X_2 F_{30}(Y) + X_2^3 F_{31}(Y) + X_2^{3\cdot 9064} F_{\lambda_1}(Y) + \dots . \tag{4.35}$$

We now examine the outer form of the inner Stokes solution so that we can match our inner solution to the complete corner-layer equations (2.15)–(2.23).

5. Outer inertial-flow solution

As we have already discussed in § 3, the appropriate outer limit in this problem is

$$\eta = X_3 \quad \text{fixed as} \quad X_2 \to 0. \tag{5.1}$$

The inner Stokes solution we have constructed in § 4 shows that it has the following outer representation in terms of the outer variable given in (5.1):

$$u \sim f_0(\eta) + X_2^2 f_2(\eta) + X_2^4 f_4(\eta) + \dots, \tag{5.2}$$

$$\phi \sim X_2\{\tfrac{1}{8}A_1\eta + \tfrac{1}{2}D_1\eta^{\frac{1}{2}(\kappa_1-1)}\sin(\kappa_2\ln\eta+\epsilon_0) + \dots\} + O(X_2^3), \tag{5.3}$$

where for $\eta \ll 1$

$$\left.\begin{aligned} f_0 &\sim A_1\eta + \{-A_{31} + \tfrac{1}{360}A_1^2(4/\pi - 21)\}\tfrac{1}{3}\eta^3 + \dots, \\ f_2 &\sim \{A_{31} - \tfrac{1}{360}A_1^2(16/\pi + \tfrac{3}{2})\}\eta + \dots, \\ f_4 &\sim -\tfrac{1}{40}A_1^2(1/\pi - \tfrac{1}{8})\,1/\eta + \dots, \end{aligned}\right\} \tag{5.4}$$

ϵ_0 being a constant, $\kappa_1 = 5\cdot 8082$ and $\kappa_2 = 1\cdot 4639$. The expression for ψ can be deduced from (5.2) by using (4.3). Now from the form of solutions given in (5.2) and (5.3) we can easily confirm that the convective term and the diffusion term of (2.15) and (2.16) are of the same order in this outer limit:

$$L_i(u_0) = O\{N_i(u, v, w)\} \quad \text{for} \quad \eta \text{ fixed as} \quad X_2 \to 0 \quad i = 1, 2. \tag{5.5}$$

This relation is certainly true for η and X_2 fixed, namely within the whole corner-layer region. This confirms our previous statement that any point on $X_2 = 0$ with finite X_3, that is, along the flow symmetry line, must also belong to the outer inertial-flow region.

A correct solution for the complete corner-layer problem in (2.15)–(2.23) must reduce to the present form of solution (5.2) and (5.3) for X_3 small as $X_2 \to 0$, the inner form of the solution in turn having already satisfied the Stokes slow-flow solution. Any corner-layer solution which does not reduce to the present form (5.2) and (5.3) in the limit of (5.1) is incorrect. We show, for example, in the appendix that Zamir's (1970) corner-layer solution on the plane $X_2 = 0$ does not reduce to the present form and hence is not a correct solution here.

The form of the expansion in (5.2) and (5.3) seems to suggest at first that a complete corner-layer solution can also be constructed in a power-series form; this is of course a familiar technique in the 'parabolic' boundary-layer theory. However, we encounter a difficulty immediately because the governing equations here are elliptic. $f_0, f_2, \ldots$ cannot be fully determined because, in determining f_0, the equation for f_0 involves the next term of the expansion f_2, unknown at this stage, and so forth. Van Dyke (1966) and his associates have developed a method of series truncation to deal with the difficulty of the present situation. If we adopt this method, the first approximation to f_0, obtained by neglecting the effect of f_2, can be fully determined and, in fact, reduces to Zamir's (1970) problem (see appendix for details of Zamir's problem). However, Zamir's solution has the wrong form in the present outer limit of (5.1). All this suggests that great care seems to be necessary in applying the series-truncation method as the starting solution itself has a wrong form which must evidently be corrected as higher order effects such as $f_2, \ldots$ are taken into account.

Besides a modified Oseen method which is explained later in the section, the series-truncation method, which, as we have just discussed, is mathematically of doubtful nature, seems to be the only analytical method available to attack the complete corner-layer problem, namely the problem for all finite values of X_2 and X_3. The numerical method of Rubin & Grossman (1969), which involves the Gaus–Seidel method of integration, seems to be the best procedure.

Now, besides the boundary conditions of (2.20), (2.21) and (2.22), the corner-layer solution must satisfy the form of solution given by (5.2) and (5.3) as $X_2 \to 0$:

$$(u, v, w)_C \to (u, v, w)_{O.S} \quad \text{with } X_3 \text{ fixed small as } X_2 \to 0, \tag{5.6}$$

where $(u, v, w)_{O.S}$ means the outer representation of the Stokes solution given by (5.2) and (5.3). In fact the addition of (5.5) to (2.20)–(2.22) as the boundary conditions implies that the boundary conditions along the four complete corners of the problem, namely $X_3 = 0$, $X_3 \to \infty$, $X_2 = 0$, $X_2 \to \infty$ are specified, which is consistent with the elliptic equations. In this way, we now see that the solution for the singular Stokes flow region is now fully satisfied and it seems that a regular numerical integration with uniform step size can now be applied over the whole domain of interest.

A question arises as to why the present outer expansion does not lead to a linearized equation as in the outer Oseen expansion for a finite-body analysis of, say, Proudman & Pearson (1957). An important difference can be traced to the geometry of the boundaries, which do not provide a length scale in this problem. The linearization of the problem from a uniform stream has no physical meaning as an approximation for low Reynolds numbers because, unlike a finite-body problem, disturbances from the body do not vanish in the outer limit. This point has been discussed by Lagerstrom (1964, p. 90). An identical situation arises in viscous flow past a semi-infinite flat plate when the Stokes solution near the leading edge is to be joined onto an outer inertial-flow solution. A meaningful linearization from the uniform flow in this inertial region may still be possible even with such problems if correctly approached. In studying a flat-plate problem of the above form Yoshizawa (private communication) has shown that two leading undetermined constants in a Stokes solution can be determined by matching with the modified Oseen solution of Lewis & Carrier (1949), with surprisingly good agreement with the existing numerical solution. A crucial factor is that the matching is carried out along the flow symmetry line as in this problem. The full significance of Yoshizawa's findings will be discussed in detail in Tokuda (1972). Despite the three-dimensionality of the flow, the present problem is remarkably similar to the flat-plate problem in flow structure and this approach will also be pursued in a future study.

6. Results and discussion

In the present paper we have constructed a singular Stokes-flow solution in the inner corner domain of the corner-layer region. An inner Stokes solution is obtained by successively applying the inner limit of X_3/X_2 finite as $X_2 \to 0$ (see equation (3.1)) and its outer expansion is obtained by applying the outer limit of X_3 finite as $X_2 \to 0$ (see equation (3.2)). This outer form of the inner solution provides an appropriate asymptotic form of the solution which the complete corner-layer solution must satisfy as $X_2 \to 0$ as well as equations (2.20)–(2.23). As far as the author is aware, no other singular flow region exists in the whole domain of X_2 and X_3 finite. Despite the simple geometry involved in the problem, the present flow seems to have a very complicated flow structure as characterized by a separating flow profile and the appearance of viscous eddies near the corner, as discussed further below.

The inner Stokes solution shows that the skin-friction coefficient based on the streamwise velocity u is given by

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 2A_1 z + (2A_1^2/15\pi)\, z^5 \ln z + 4A_{31} z^5 + \ldots. \tag{6.1}$$

For the purpose of comparison with the experiments of Zamir & Young (1970) and also Rubin & Grossman (1969), the distance measured in the original Cartesian co-ordinate system is used in (6.1). Zamir & Young's experiments and

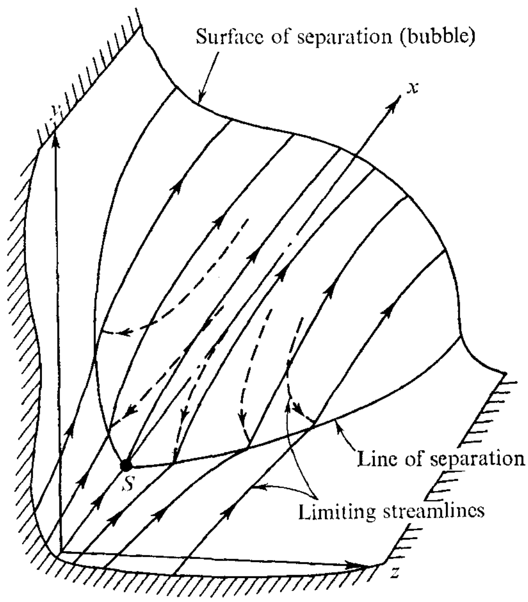

FIGURE 4. Maskell's model for three-dimensional flow separation.

Rubin & Grossman's numerical results all confirm that the skin friction is of linearly vanishing type near $z = 0$ or $X_2 = 0$, as in (6.1). Therefore $A_1 \neq 0$. Hence the F_1 term does not vanish and consequently the logarithmic term of $O(X_2^{\frac{3}{2}} \ln X_2)$ must intervene in the second approximation for the streamwise velocity. The appearance of a separating velocity profile is always a warning for a possible singular flow behaviour at such a point. The present asymptotic solution is valid only far downstream as $X_1 \rightarrow \infty$. Therefore in practice a flow separation would not take place. Interesting information may still be obtained on the separation flow pattern of this configuration from the dominant term of the present solution (4.11), supposing that such a separation is induced at point S of figure 4 by some mechanism, say by a slight adverse pressure gradient. It may be confirmed from (4.11) that the point S is an only singular point involved in the flow field. Therefore, according to Maskell's (1955) classification, the separated flow pattern would take on a separation bubble form as illustrated in figure 4. The flow near $X_2 = 0$, which is obviously singular, seems to require careful analysis for this reason.

In the cross-flow plane, the leading term of the Stokes solution is a simple radial flow distribution which arises from ψ_1 term. A most interesting result which our analysis has unveiled is the appearance of a sequence of viscous eddies in the cross flow. This will dominate the terms of $O(X_2^{\frac{3}{2}} \ln X_2)$ and $O(X_2^{\frac{3}{2}})$, which are induced partly by the second-order streamwise flow u_3 and partly by the nonlinear effects of the problem. In the inertial-flow region the viscous-eddy solution has the form given in (4.17) (see also equation (5.3)). The numerical solution of Rubin & Grossman (1969), which is intended for the analysis in an inertial-region, does not show viscous eddies in the inner corner-layer region although it does show some tendency for closed vortical patterns right in the corner. The experimental evidence of Zamir & Young (1970) is not decisive on this point. In examining their visualization picture of gas traces near the corner junction, we see that those traces near the intersection always tend to blur more widely than those away from it. To the present author, this seems to imply that gas near the corner is trapped in one of several small eddies, thus demonstrating the existence of the eddies. Because this Stokes region in which such an eddy solution appears is a singular region inbedded within the corner-layer region, much care seems to be required to reconstruct an eddy solution by numerical integration. Indeed Burggraf (1966) has reconstructed a viscous-eddy solution for a two-dimensional square cavity flow consistently up to Reynolds numbers of 1000 with a carefully calculated numerical solution. A Reynolds number of this problem based on a typical velocity in the cross-flow is obviously of $O(1)$ although that based on a streamwise velocity is large. Therefore, as far as the cross-flow is concerned, Burggraf's numerical solution seems to provide interesting insight to our problem. In all probability, a viscous-eddy flow exists. Another interesting result from Burggraf's solution is that an almost inviscid flow region exists at the centre of the cavity within an inertial-flow region, with the boundary-layer type regions appearing near the cavity wall. This shows that in this problem an almost inviscid-flow region with low shear flow region exists, particularly along the flow symmetry line with the boundary-layer type viscous effect concentrated either near the free surface or away from the symmetry line. Zamir & Young (1970) have observed exactly this flow pattern in their experiments. The bulge in the isovels which they observed across the flow symmetry line implies that a low shear flow region there is surrounded by a high shear region. Because of the singularity involved near a sharp corner, a finite-difference method will never probe the flow field fully. A mathematical representation such as the present asymptotic analysis is vital if an adequate operational procedure for solution is to be given (see Motz 1946).

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Appendix. Zamir's (1970) solution on $X_2 = 0$

In studying the present corner-layer solution valid on the surface of the flow symmetry $X_2 = 0$, Zamir (1970) has noted a similarity solution and deduced the following ordinary differential equation by assuming that the boundary-layer type approximations are valid across this plane, namely $\partial^2 u/\partial X_2^2 \ll \partial^2 u/\partial X_3^2$.† We have

$$\eta^6(f_0''' - f_0 f_0'') + 2\eta_0^3(f_0 f_0' - 2f_0'') + 2\eta_0(4f_0' - f_0^2) - 8f_0 = 0, \tag{A 1}$$

† This approximation actually corresponds to the first approximation of the series-truncation method in which the effect of f_2 on f_0 is ignored.

where $\eta_0 = X_3^{\frac{1}{2}}$ and u and w are related to f_0 by the continuity equation:

$$u = f_0' - 2f_0/\eta_0, \quad w = \tfrac{1}{2}X_1^{-\frac{1}{2}}(\eta f_0' - f_0), \quad \text{with} \quad v = 0 \quad \text{on} \quad X_2 = 0.$$

For $\eta_0 \ll 1$, a solution which satisfies the wall condition can be developed as

$$f_0 \sim a\left(\eta_0^4 + \frac{3}{7.5.4}a\eta_0^9 + \frac{31}{13.5^2.4^2.7}a^2\eta_0^{14} + \ldots\right),$$

where a is an arbitrary constant. The streamwise velocity on $X_2 = 0$ is therefore

$$u \sim a\left(2X_3^{\frac{3}{2}} + \frac{3a}{5.4}X_3^4 + \ldots\right). \tag{A 2}$$

If the leading term $X_3^{\frac{3}{2}}$ is to be a correct solution, one must be able to find the harmonic solution in the inner Stokes region in the present inner limit. Therefore we write as the leading term of the inner streamwise velocity u

$$u_1 = X_2^{\frac{3}{2}} F_1(Y),$$

where

$$(1+Y^2)F_1'' - YF_1' + \tfrac{3}{4}F_1 = 0. \tag{A 3}$$

A general solution of (A 3) can be written down by the contour integration as

$$F_{1i} = \int_{C_i} \frac{(\zeta - z)^{\frac{3}{2}}}{1+\zeta^2}\, d\zeta \quad \text{for} \quad i = 1, 2, \tag{A 4}$$

where C_i is such that $(\zeta - z)^{\frac{1}{2}}|_{C_i} = 0$.

Two independent solutions are obtained if C_1 includes i and $C_2 - i$ within the contours. From this, we can deduce the correct solution satisfying the no-slip condition on the wall and also the flow symmetry condition is

$$F_1 = A_1(1+Y^2)^{\frac{3}{4}} \sin\{\tfrac{3}{2}\tan^{-1}|Y|\}. \tag{A 5}$$

The absolute sign in $\tan^{-1}|Y|$ is necessary to ensure the flow symmetry. Although (A 5) provides the leading term of the expansion $X_2^{\frac{3}{2}}$ in the outer representation, the derivatives of further terms show a discontinuity across the line $X_2 = 0$. We must conclude, therefore, that (A 1) is not a correct equation for the corner-layer flow on $X_2 = 0$. In fact we have shown in § 5 that $\partial^2 u/\partial X_3^2$ must be retained even on the plane $X_2 = 0$. In that case no similarity solution such as that noted by Zamir (1970) could be found. This confirms that the corner-layer solution must be consistent with the inner Stokes solution in the overlap domain of X_3 finite as $X_2 \to 0$.

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